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String Theory

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1 Action Principals

1.1 The Relativistic point particle

We begin our discussion with the model of a point particle that is subject to relativistic effects. Such a particle traces out a line in spacetime known as the worldline. We know that the length of this world line is given by the equation

$$ds^2 = -dX^\mu g_{\mu\nu} dX^\nu \tag{1}$$

Where μ and $\nu = 0, 1, 2, ..., D$. D is the dimension of spacetime $g_{\mu\nu}$ is the spacetime metric tensor and 'ds' is the proper length. The simplest lorentz invarient action we can write for this particle would be proportional to the proper length

$$S = -mc \int ds \tag{2}$$

Where 'mc' is here for dimensional purposes. We can then write

$$\begin{aligned} S &= -mc \int \sqrt{-dX^\mu g_{\mu\nu} dX^\nu} \\ &= -mc \int \sqrt{-\frac{dX^\mu}{dt} dt g_{\mu\nu} \frac{dX^\nu}{dt} dt} \end{aligned} \tag{3}$$

If we assume that spacetime is flat then we write

$$\begin{aligned} S &= -mc \int dt \sqrt{-\frac{dX^\mu}{dt} \eta_{\mu\nu} \frac{dX^\nu}{dt}} \\ &= -mc \int dt \sqrt{-\frac{dX^\mu}{dt} \frac{dX_\mu}{dt}} \end{aligned} \tag{4}$$

Now if we contract the 0th term we can simply the action to the following quantity

$$\begin{aligned} &= -mc^2 \int dt \sqrt{1 - \frac{dX^i}{dt} \frac{dX_i}{dt}} \\ &= -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \tag{5}$$

Where v^2 is the D-dimensional velocity squared. We can easily show that this reduces to the classical lagrangian in the non-relativitic limit.

$$\begin{aligned} L &= -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \\ &\approx -mc^2 (1 - \frac{1}{2} \frac{v^2}{c^2}) = \frac{1}{2} mv^2 + \text{constant} \end{aligned} \tag{6}$$

So it reduces to the classical result in the non relativitic limit.

1.2 World Sheets and Nambu–Goto action

Now let us consider modeling a relativistic string. The simplest action we can construct would be proportional to the proper area traced out in spacetime by the string. This is called the Nambu-Goto action. We will begin by considering the proper area in General Relativity.

In General Relativity the proper area is given by the corresponding formula.

$$A = \int d\tau d\sigma \sqrt{-\det(g_{ab})} \tag{7}$$

Here we have chosen Tau and sigma as parameterizations. Tau does not necessarily represent the proper time. Now there are a few caveats we need to take care of later. First lets motivate this result by consider breaking up the 'world sheet' into rectangles. Seeing how we are working with vectors, it makes sense to break it up into parallelograms by considering the identity in linear algebra.

$$dA = |dv_1 \times dv_2| = |dv_1||dv_2||\sin(\theta)| = dv_1||dv_2||\sqrt{1-\cos^2(\theta)|} \tag{8}$$

$$dA = \sqrt{|dv_1|^2|dv_2|^2 - |dv_1|^2|dv_2|^2\cos^2(\theta)|} \tag{9}$$

Now we know that dv1 and dv2 should be in different directions. We shall write this in terms of the tensor function $X^\mu(\tau, \sigma)$ that represents the parametrized world-sheet. We will assign the value dv1 to $\frac{dX^\mu}{d\tau}d\tau$ and dv1 to $\frac{dX^\mu}{d\sigma}d\sigma$ where sigma and tau are used to parametrize the world sheet. Now this prodcues the following result

$$\begin{aligned} dA &= \sqrt{(d\tau^2\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau})(d\sigma^2\frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma}) - d\tau^2 d\sigma^2(\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma})^2} \\ &= d\tau d\sigma\sqrt{(\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau})(\frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma}) - (\frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma})^2} \end{aligned} \tag{10}$$

Now, if we believe equation (7) then we should be able to express equation (10) as the determinant of some metric. By inspection we can write an induced metric 'h' as the following.

$$\begin{aligned} h_{\alpha\beta} &= \begin{pmatrix} \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\tau} & \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma} \\ \frac{dX^\mu}{d\tau}g_{\mu\nu}\frac{dX^\nu}{d\sigma} & \frac{dX^\alpha}{d\sigma}g_{\alpha\beta}\frac{dX^\beta}{d\sigma} \end{pmatrix} \\ &= \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} \\ &= \partial_\alpha X^\mu \partial_\beta X_\mu \end{aligned} \tag{11}$$

And all together the Nambu-Goto action becomes.

$$S = -T\int d\tau d\sigma\sqrt{-\det(h_{\alpha\beta})} \tag{12}$$

Where the scaler '-T' is introduced for dimensional purposes. And the negative inside the square root comes as a consequence of the Pseudo-Riemannian space which ensures the determinant will be negative.

Equation (11) is the known as the induced metric. It shouldn't come as a surprise that we are now working with two metrics, since our strings are manifolds themselves. It also shouldn't be a surprise that the metric can be expressed as a 2x2 matrix since our world-sheet traces a two dimensional surface. The second equal sign in equation (11) can be easily checked to be true.

Now there is actually a problem with this action. The problems stems from renormalization, namely, the square root makes renomrlaization difficult. Correcting this problem requires us to rewrite the Nambu-Goto action using what is known as the auxillary world sheet metric. It is a metric that classically reduces the action to the Nambu-Goto action. This new action in terms of the auxiliary world sheet metric is known as the Polyakov action which is given by

$$S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{13}$$

Where the $\gamma^{\alpha\beta}$ represents the auxiliary world sheet metric with both indices raised and the $\gamma = \det(\gamma_{\alpha\beta})$. This corrects the issue and makes the theory renormalizable. The scaler at the begining is there for dimensional purposes.

We now wish to show that this indeed reduces to equation (12) at the classical level. To show this we will find variation of the action with respect to the inverse world-sheet metric $\gamma^{\alpha\beta}$

$$\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0 \tag{14}$$

To accomplish this, we begin by finding the variation of equation (13)

$$\begin{aligned}\delta S &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu) \\ &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}\delta(\gamma^{\alpha\beta})\partial_\alpha X^\mu \partial_\beta X_\mu\end{aligned} \tag{15}$$

Using the following identity

$$\delta\gamma = -\gamma\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} \tag{16}$$

Which in our case implies that following relationship

$$\delta\sqrt{-\gamma} = -\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} \tag{17}$$

Going back to equation (15) its easy to see that

$$\begin{aligned}\delta S &= \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}\delta(\gamma^{\alpha\beta})\partial_\alpha X^\mu \partial_\beta X_\mu \\ &= \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta\gamma^{\alpha\beta}\frac{1}{2}\sqrt{-\gamma}\gamma_{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta\gamma^{\alpha\beta}(-\gamma)^{\frac{1}{2}}\partial_\alpha X^\mu \partial_\beta X_\mu \\ &= 0\end{aligned} \tag{18}$$

Dividing equation (18) by $\delta\gamma^{\alpha\beta}$ and canceling out the constants gives the equation of motion for the metric. In other words, it's easy to see that $\frac{\delta S}{\delta\gamma^{\alpha\beta}} = 0$ implies the following relationship.

$$\frac{1}{2}\gamma_{\alpha\beta}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu = \partial_\alpha X^\mu \partial_\beta X_\mu \tag{19}$$

From here we can take the negative square root of the determinant from both sides wrt to the alpha and beta tensor,

$$\frac{1}{2}(-\gamma)^{\frac{1}{2}}\gamma^{\kappa\eta}\partial_\kappa X^\mu \partial_\eta X_\mu = \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} \tag{20}$$

But the right side is just the integrand of the Nambu-Goto action and the left side is the integrand of the Polyakov action. Therefore the equation of motion for the world sheet metric $\gamma_{\alpha\beta}$ implies the Nambu-Goto action. QED.

Now notice that we never explicitly defined a value for the world-sheet metric $\gamma_{\alpha\beta}$. This is to preserve the symmetries in the Polyakov action. Namely, these symmetries allows us to perform gauge fixing with the only requirement being that equation (19) is satisfied. We will now explore these symmetries in the next section to see which quantities we are allowed to gauge fix.

1.3 Symmetries in the Polyakov action

We would now like to talk about the symmetries in the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{21}$$

...for which there are a lot. However, for now, lets restrict our analysis to that of Minkowski space. The Polyavok action is invarient under the following transformations

Poincare Transformation

$$X'^\mu = \Lambda^\mu_\nu X^\nu + a^\mu \tag{22}$$

$$\delta\gamma_{\alpha\beta} = 0 \tag{23}$$

Where Λ is a Lorentz transformation and a a^μ is a translation. Poincare invariance is a global symmetry in the action which implies it cannot be used for gauge fixing.

Diffeomorphism invariance

$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma) \tag{24}$$

$$\frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma) \tag{25}$$

for some new choice of coordinates $\sigma'^a(\tau, \sigma)$. Equation (24) is also known as reparameterization invariance. This is an very important property of string theory so be sure to understand it. While equation (25) is nothing more than the tensor transformation law. Diffeomorphism invarice is a local symmetry which implies it allows for gauge fixing.

Weyl invariance

$$X'^{\mu}(\tau,\sigma) = X^{\mu}(\tau,\sigma) \tag{26}$$

$$\gamma'_{ab}(\tau,\sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma) \tag{27}$$

which holds for any $\omega(\tau,\sigma)$. This is also a local gauge symmetry, which implies that it can be used for gauge fixing.

This concludes the relevant symmetries in the Polyavok action. While in the future, we will talk more about the implication of these symmetries and treat them as more fundamental then the Polyakov action, for now they are simply symmetries in the Polyakov action.

1.4 Equations of motion and Boundary conditions

The goal of this section is find the equation of motion for X^{μ} in the Polyakov action.

$$S[X^{\mu},\gamma_{\alpha\beta}] = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} \tag{28}$$

We will now take the variation wrt to X^{μ}. Where the goal is to find equation of motion by demanding that

$$\frac{\delta S}{\delta X^{\mu}} = 0 \tag{29}$$

Taking the variation

$$\begin{aligned}\delta S &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\delta((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu}) \\ &= -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} - \frac{1}{4\pi\alpha'}\int_M d\tau d\sigma\ (-\gamma)^{\frac{1}{2}}(\gamma^{\alpha\beta})\partial_{\alpha}X^{\mu}\partial_{\beta}\delta X_{\mu}\end{aligned} \tag{30}$$

Where we used the easy to show identity $\delta\partial_a X^{\mu} = \partial_a\delta X^{\mu}$. By relabeling the indices we can combine them into a singal integral. (Specifically, we use the symmetry in the metric tensor)

$$\delta S = -\frac{1}{2\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} \tag{31}$$

Now from here we manipulate the integrand using the identity

$$(-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\alpha}\delta X^{\mu}\partial_{\beta}X_{\mu} = \partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\delta X^{\mu}\partial_{\beta}X_{\mu}) - \delta X^{\mu}\partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\partial_{\beta}X_{\mu}) \tag{32}$$

Where this can be understood to be an integration by parts. Note that distributing the derivatives on the right side will equal the left side. Now, you may be familiar that the term

$$\partial_{\alpha}((-\gamma)^{\frac{1}{2}}\gamma^{\alpha\beta}\delta X^{\mu}\partial_{\beta}X_{\mu}) = 0 \tag{33}$$

is usually understood to vanish since it is a total derivative inside the action. Actually it vanishes due to the boundary conditions on the string, but we will just assume it to be zero for now

From here, we are left with the action (after dividing by δX^{μ})

$$\frac{\delta S}{\delta X^{\mu}} = \frac{1}{2\pi\alpha'} \int_{M} d\tau d\sigma \partial_{\alpha}((-\gamma)^{\frac{1}{2}} \gamma^{\alpha\beta} \partial_{\beta} X_{\mu}) = 0 \tag{34}$$

The equation of motion is understood to mean the following equations

$$\partial_{\alpha}(-\gamma)^{\frac{1}{2}} \gamma^{\alpha\beta} \partial_{\beta} X_{\mu} = 0 \tag{35}$$

Along with

$$\begin{aligned} (-\gamma)^{\frac{1}{2}} \gamma^{\alpha\beta} \partial_{\alpha} \partial_{\beta} X_{\mu} &= (-\gamma)^{\frac{1}{2}} \partial_{\alpha} \partial^{\alpha} X_{\mu} = 0 \\ &= (-\gamma)^{\frac{1}{2}} \partial_{\alpha} \partial^{\alpha} X_{\nu} g^{\mu\nu} = g^{\mu\nu} * 0 \\ &= (-\gamma)^{\frac{1}{2}} \partial_{\alpha} \partial^{\alpha} X^{\mu} = (-\gamma)^{\frac{1}{2}} \nabla^{2} X^{\mu} = 0 \end{aligned} \tag{36}$$

This along with the previous section complete the equations of motion for the Polyakov action at this state. Its worth noting that there are a few terms that can be added to the Polyakov action that, although break Poincare invariance, are worth exploring. We will discuss these later.

Lets now talk about boundary conditions. There are two types of strings that are constructed from various boundary conditions. The two types are open strings and closed strings. Intuitively, we can think of closed strings as being topologically a circle and an open string as being topologically a line interval. Let us now discuss these boundary conditions.

For these conditions we will choose a parametrization for σ such that it lies inside the interval $0 \leq \sigma \leq \pi$. This is an arbitrary choice that is done for the sake of simplifying the analysis of the boundary conditions.

Closed String

$$X^{\mu}(\tau, \sigma) = X(\tau, \sigma + \pi) \tag{37}$$

This is a period condition that simply ensures the string is closed everywhere.

Open String with Neumann boundary conditions

$$\frac{\partial X^{\mu}}{\partial \sigma} = 0 \text{ At } \sigma = 0, \pi \tag{38}$$

This consequence of this condition is that the component of momentum normal to the worldsheet vanishes at the boundary.

Open String with Dirichlet Boundary condition

$$X^{\mu}|_{\sigma=0} = X_{0}^{\mu} \tag{39}$$

$$X^\mu|_{\sigma=\pi} = X^\mu_\pi \tag{40}$$

Where X^μ_0 and X^μ_π are constants. This condition applies for $\mu = 1, 2, 3..D - p - 1$ where d is the dimension of the theoy p is a p-dimensional subspace of the theory. We will talk more about Dp branes later.

1.5 Light Cone coordinates

The purpose of this section construct an easier set of coordintates to solve the equations of motion. These are known as light cone coordinates and they are merlely a new set of coordinates to makes solving our theory easier. In General relativity. we define the light cone component for any vector a^μ as the following

$$a^+ \equiv \frac{1}{\sqrt{2}}(a^0 + a^1) \tag{41}$$

and

$$a^- \equiv \frac{1}{\sqrt{2}}(a^0 - a^1) \tag{42}$$

we let the rest of the indices run from i = 2,...,D

$$a^i \text{ runs from i = 2, . . . , D} \tag{43}$$

We can also define coordinates with 'lowered' as the following

$$a_+ \equiv -a^- \tag{44}$$

and

$$a_- \equiv -a^+ \tag{45}$$

These new coordinates are usually examined in a special frame such that the contractions are done as follows

$$\begin{aligned} a^\mu b_\mu &= -a^+ b^- - a^- b^+ + a^i b^i \\ &= a_- b^- + a_+ b^+ + a_i b^i \end{aligned} \tag{46}$$

Where we choose metric such that $a_i = a^i$

Back to string theory, we introduce world-sheet light cone coordinates defined by

$$\sigma^\pm = \tau \pm \sigma \text{ and } \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{47}$$

Along with the metric

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{48}$$

We will now discuss solutions to the Equations of motion

2 Conformal Field Theory