Numerical solutions to Pendulum equations

The purpose of this paper is to explore the various graphs produced by several different equations that describe the motion of a pendulum. Most of these equations do not contain analytical solutions, therefore, it is necessary to employ the use of numerical methods to obtain the general shape of their trajectory. The algorithm used to accomplish this is RK4.

Derivation of the equation

Newtonian mechanics can be used to derive the Pendulum equation. We begin with Newton's second law.

$$\Sigma F_i = ma \tag{1}$$

Note that equation (1) is strictly for cartesian coordinates. For our analysis, it is worth rewriting equation (1) to be in polar coordinates which is much more natural to our problem.

$$\Sigma F_r = m(\frac{d^2 r}{dt^2} - r\frac{d\phi}{dt}) \tag{2}$$

$$\Sigma F_\phi = m(r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt}) \tag{3}$$

For equation (2) and equation (3), 'r' represents the 'radius axis' which is in the direction of the point particle from the place the string is tied. The phi vector represents the axis tangential to the point particle. In our case, the point particle is constrained to a circular motion since it is acting as a pendulum. Consequently, the first equation (2) isn't as interesting since the object will remain at a constant radius from the origin. The second equation (3) can be rewritten using the following relationship.

$$\Sigma F_\phi =- mgsin(\phi) \tag{4}$$

Specifically the sine component of the gravitational force will give us the net tangential force. Note that the string tension will not impact the net tangential force because the tension is a constraint force. Replacing equation (4) into equation (3) gives the following equation.

$$- mgsin(\phi) = m(r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt}) \tag{5}$$

However, the radius is constant therefore the second term cancels out.

$$- gsin(\phi) = r\frac{d^2\phi}{dt^2} \tag{6}$$

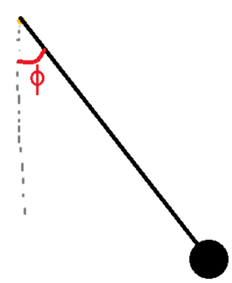
$$\frac{d^2\phi}{dt^2} = -\frac{g}{r}sin(\phi) \tag{7}$$

Equation (7) is typically written in terms of the string length which is equal to the radius. Therefore, the following relationship holds

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}sin(\phi) \tag{8}$$

Equation (8) is the equation of interest. It describes the angle made from an object in a pendulum like motion measured from the negative y-axis towards the positive x-axis. The following image demonstrates this specific angle (phi).

Image 1A



Note that several approximations can be made that allow us to obtain an approximate analytical solution for the pendulum. The simplest one is the small angle approximation that assumes the following relationships.

$$sin(\phi) \cong \phi \tag{9}$$

Equation (9) holds for small values of phi. This approximation allows us to analytically solve equation (8). This produces the following result. (Assuming the initial angular velocity is zero)

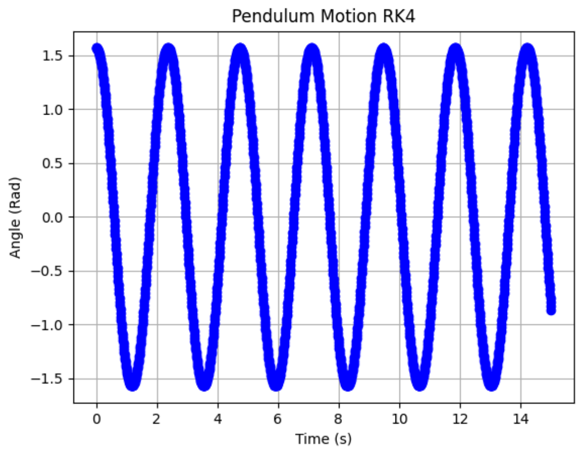
$$\phi(t) \cong \phi_0 cos(t\sqrt{\frac{g}{L}}) \tag{10}$$

Where phi initial is the initial angle We will compare the solutions provided by this approximation to the numerical ones solved via computational methods.

Graphs found for RK4 and small angle approximation

To start, a small angle of pi/5 is used on the RK4 numerical computation and compared to the small angle approximation. The gravitational constant was assumed to be 9.80665 m/s^2 and the length of the pendulum was assumed to be 1 meter. The following graph shows the results for the angle as a function of time.

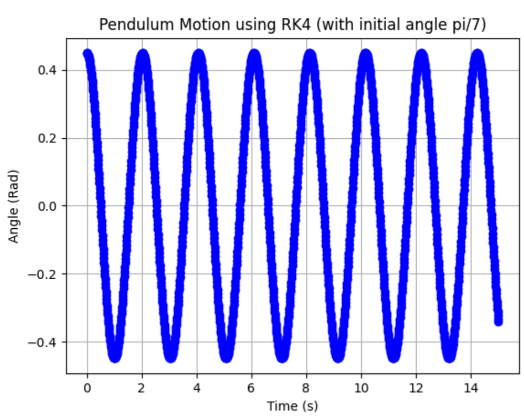
Graphs 1A

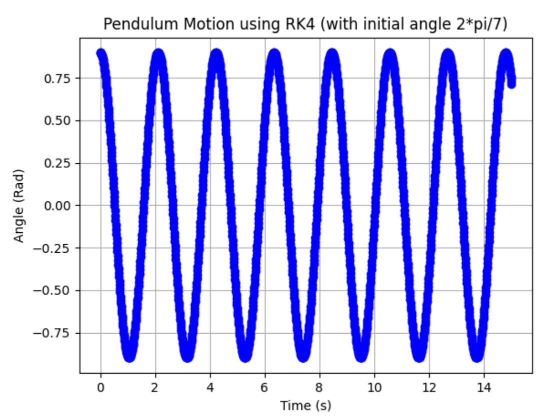


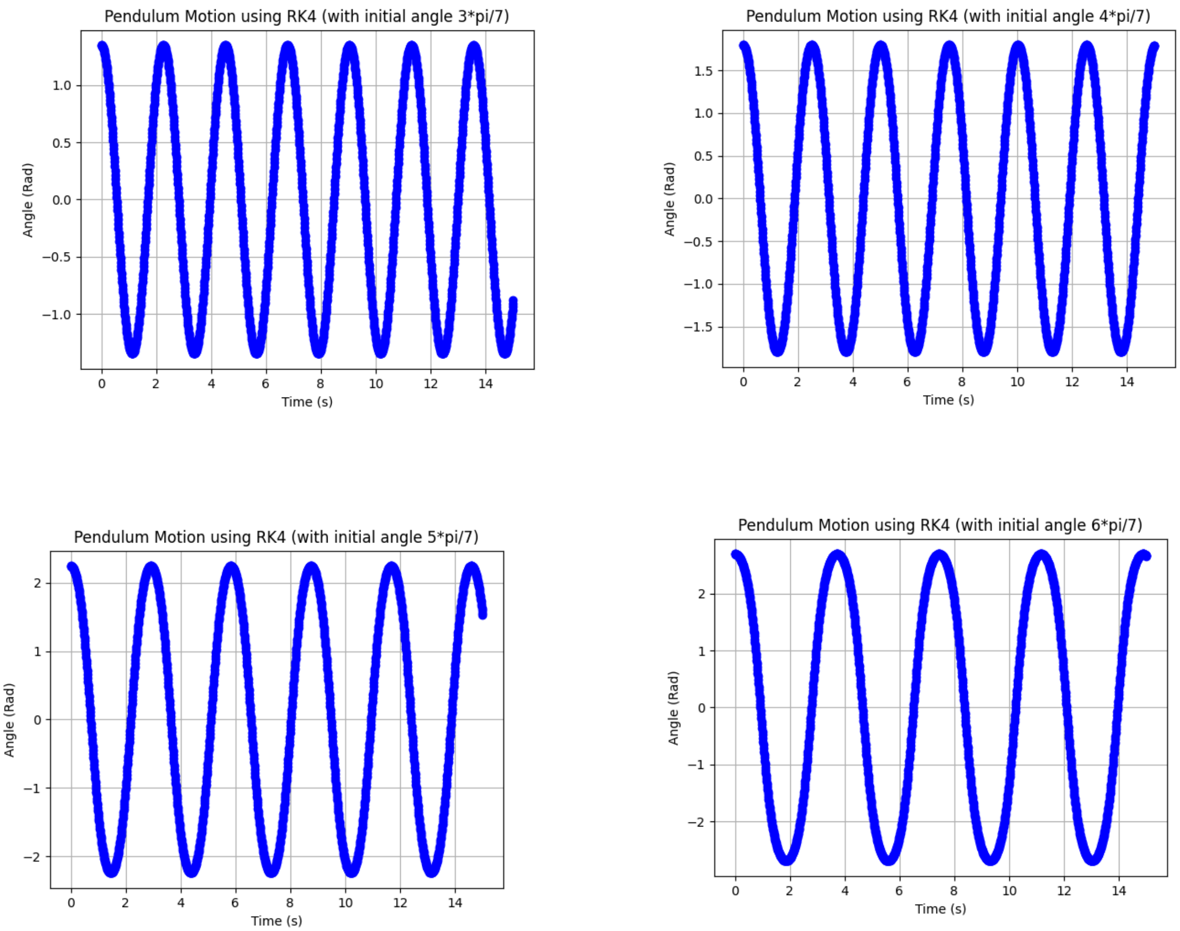
Graph 1A depicts the motion of a pendulum for about 14 seconds. RK4 is used with a gravitational constant of 9.80665 m/s^2 and a pendulum length of 1 meter.

Examining the graph, it is clear that the graph resembles a sinusoidal graph for small angles. This agrees with the small angle approximation for short time frames. The following graphs depicts angles pi/7, 2pi/7, 3pi/7, 4pi/7, 5pi/7, 6pi/7 as a function

Graphs 1B





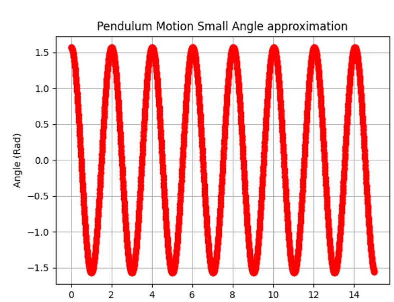


*Graphs 1B depicts plots of the RK4 method with the initial angle pi/7 to the angle 6*pi/7 in increments of pi/7. The length of the pendulum was fixed at 1 meter and the gravitational constant was set to 9.80665 m/s^2*

Examining the graphs, it is clear that the period seems to be increasing as the starting angle is increased. It is also clear that latter graphs no longer resemble a sinusoidal pattern. This makes sense since the small angle approximation should only hold for small angles.

Next we decided to

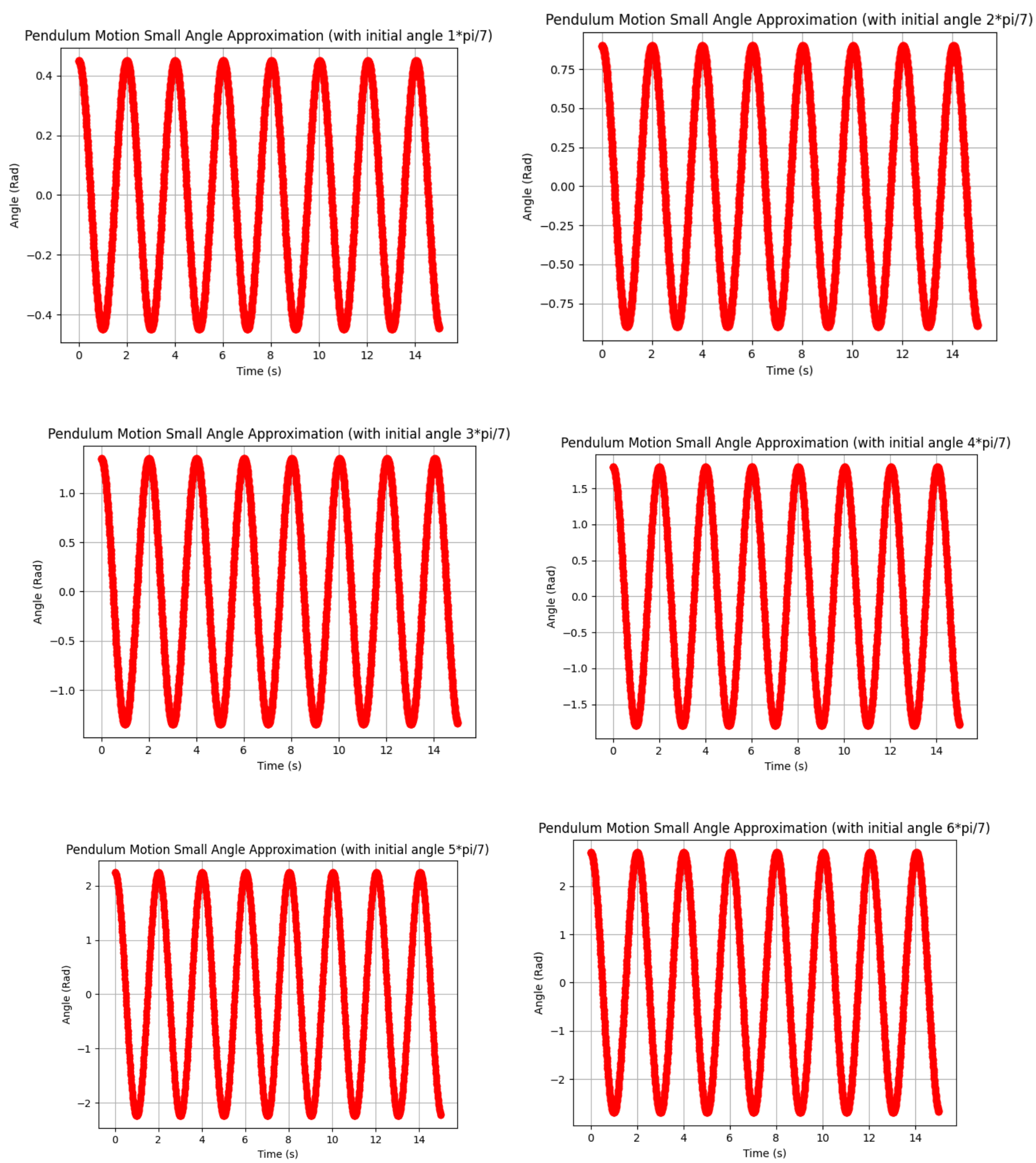
Graphs 2A



Graph 2A depicts the angle as a function of time found from equation (10) with the initial angle of pi/5.

Graphs for higher angles can be evaluated and compared to the numerical graphs. Note that the small angle approximation only truly holds for small angles. Therefore, the following graphs are merely a testament of the inaccuracy of the small angle approximation for higher angles.

Graph 2A

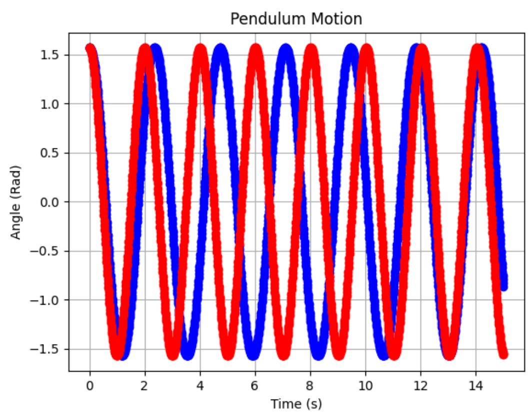


*Graphs 2B depicts plots of the small angle approximation with the initial angle pi/7 to the angle 6*pi/7 in increments of pi/7. The length of the pendulum was fixed at 1 meter and the gravitational constant was set to 9.80665 m/s^2*

Here it is clear that the small angle approximation no longer matches the RK4 method for higher angles. This is clear since the period of the small angle approximation increases for higher angles.

Overlapping both graphs 1A and 2A produces the following graph

Graph 3A



Graph 3A depicts the overlap of the small angle approximation (red graph) and the RK4 graph (blue graph) for an initial angle of pi/5

Examining the graph, it is clear that the small angle approximation will diverge from the RK4 numerically found. This highlights the importance of using numerical methods instead of analytic approximations. As shown in the graph, the graphs and movement of the pendulum will no longer match after a few seconds. Indeed, numerical methods can help us gain a clearer picture of the theory because they will remain far more accurate compared to the small angle approximation.

Another analysis that can be accomplished is computing the period of the numerical pendulum. Physically, we would expect that the period would increase as a function of the initial angle. We can see this using our simulation with a varying period.

This was accomplished by finding the time difference between two non consecutive zeros using a 0.09 radians and -0.09 radians threshold to avoid floating point comparison.

Table 1

Initial Angle	Numerically Found Period
$\frac{\pi}{7}$	$2.030s$

$\frac{2\pi}{7}$	$2.110\mathbf{s}$
$\frac{3\pi}{7}$	$2.260\mathbf{s}$
$\frac{4\pi}{7}$	$2.510\mathbf{s}$
$\frac{5\pi}{7}$	$2.920\mathbf{s}$
$\frac{6\pi}{7}$	$3.720\mathbf{s}$
$\frac{7\pi}{7}$	$0.000s$ **(N/A)**

Table 1 depicts the period found though RK4 and finding the approximate distance between crests by finding the zeros of the function (via a small threshold to avoid floating point error) and subtracting the time difference between two non consecutive zeros. Note that the length used for the pendulum was 1 meter and the gravitational constant was 9.80665 m/s^2

Indeed the data shows that the period will increase nonlinearly as the initial angle increases as a function of initial angle. Indeed, this matches our physical expectation that the period should increase with the initial angle.

On the other hand the period from the small angle approximation is found to be the following.

$$T \cong 2\pi\sqrt{\frac{L}{g}} \tag{11}$$

Note that equation (11) doesn't depend on the initial angle since this is based on the small angle approximation. Using the values from our simulation of length 1 meter and gravitational constant of 9.80665 m/s^2 produces the following value.

$$T \cong 2.006s \tag{12}$$

Indeed, this shows that the value is comparable to small angels. For instance in Table 1 the value found for the angle pi/7 is 2.030 seconds.

Air resistance model

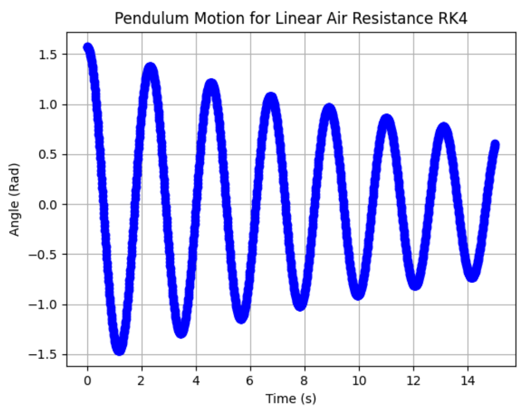
Finally we will examine the numerical graphs of a non ideal pendulum. Specifically, we will consider an air resistance term proportional to the angular velocity term.

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}sin(\phi) - \boldsymbol{\alpha}\frac{d\phi}{dt} \tag{13}$$

Equation (13) depicts the air resistance pendulum differential equation where alpha is a unitless constant that represents the coefficient of linear air resistance.

Using the test value of 0.1 with the standard gravitational constant and the length of 1 meter produced the following graph. The initial angle was set to a value of pi/2

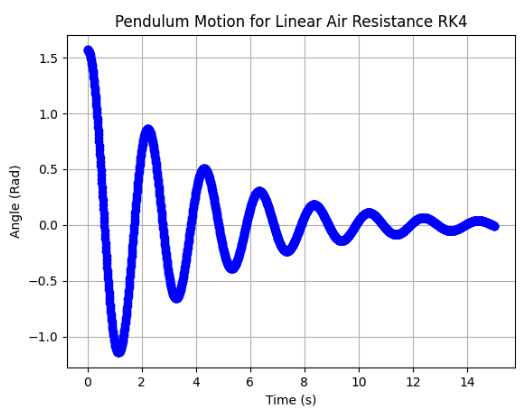
Graph 4A



Graph 4A depicts linear air resistance for alpha set to 0.1 with an initial angle of pi/2

As it’s shown, the graph seems to decay in a somewhat linear manner. The amplitude seems to decrease substantially throughout the trajectory

Graph 4B

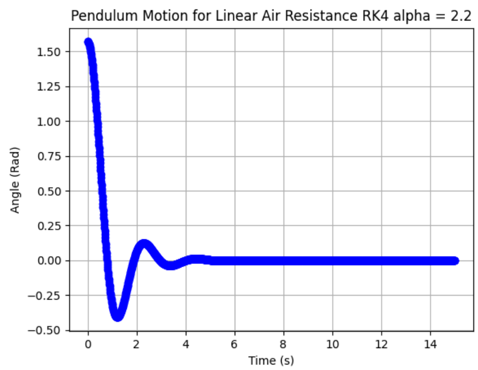


Graph 4B depicts linear air resistance for alpha set to 0.5 with an initial angle of pi/2

Here, it is clear that the decay occurs more rapidly than the previous graph. It shows the general trend towards zero. The decay also seems to be exponential as shown in the general height of the peaks.

Finally we tried an alpha value of 2.2 which produced the following graph.

Graph 4C



Graph 4C depicts linear air resistance for alpha set to 2.2 with an initial angle of pi/2

It is clear that these graphs resemble the damped and underdamped systems of an oscillator.

Overall it seems like damping the object affects will cause an exponential decay on the amplitude until it terminates. Further analysis would be required to create a more realistic model that would undergo quadratic air resistance. Moreover, several experiments would need to be conducted to determine the main source of damping on a real pendulum. In essence the RK4 method provides an excellent way to model a more realistic pendulum as opposed to the small angle approximation which seems to lose accuracy even for small angles after a few seconds.